Tougher Trig Series and the landen-dioid constant.

$$\textit{The series: } S=\sum_{n=1}^{\infty}(-1)^n\frac{|\sin(n)|}{n}, \textit{ converges.} \tag{1}$$

Proof. This series seems to be difficult. landen relied heavily on discussions of a similar series on AoPS Math Forum.

Define:

$$F(n) = \sum_{k=1}^{n}\left(|\sin(2k-1)|-\frac{2}{\pi}\right) \tag{2}$$

$$G(n) = \sum_{k=1}^{n}\left(|\sin(2k)|-\frac{2}{\pi}\right) \tag{3}$$

$$lim_{n\to\infty}\frac{F(n)}{n} = \frac{G(n)}{n}=0 \tag{4}$$

The limits (4) follow from the fact that $|\sin(2x)|$ and $|\sin(2x-1)|$ are periodic functions with irrational period $\frac{\pi}{2}$ and, therefore, the average of the sum will converge to the reciprocal of the period times the integral over the period.

$$\sum_{k=1}^{n}\frac{1}{2k-1}\left(|\sin(2k-1)|-\frac{2}{\pi}\right) = F(1)+\sum_{k=2}^{n}\frac{F(k)-F(k-1)}{2k-1} \tag{5}$$

$$= \frac{F(n)}{2n-1}+2\sum_{k=1}^{n-1}\frac{F(k)}{(2k-1)(2k+1)} \tag{6}$$

Equation (6) follow by rearranging terms of (5) or by summation by parts, which is equivalent. The first term on the right of (6) has $\lim n\to\infty=0$ by (4). It remains to show the the sum in (6) converges. Similar manipulations of $G(n)$ yield:

$$\sum_{k=1}^{n}\frac{1}{2k}\left(|\sin(2k)|-\frac{2}{\pi}\right)=\frac{G(n)}{2n}+2\sum_{k=1}^{n-1}\frac{G(k)}{(2k)(2k+2)} \tag{7}$$

In order to get a better estimate of sums involving $|\sin(x)|$ we can use its nicely convergent Fourier series and some useful trigonometric identities:

$$|\sin(x)| = \frac{2}{\pi}-\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\cos(2jx)}{4j^2-1} \tag{8}$$

$$= \frac{2}{\pi}-\frac{4}{\pi}\sum_{j=1}^{m}\frac{\cos(2jx)}{4j^2-1}+O(\frac{1}{m}) \tag{9}$$

$$\sum_{k=1}^{n}\cos((2k-1)x) = \frac{\sin(2nx)}{2\sin(x)} \tag{10}$$

$$\sum_{k=1}^{n}\cos(2kx) = \frac{\sin((n+1/2)x)}{\sin(x/2)}-\frac{1}{2} \tag{11}$$

Substituting (9) into (2) and then interchanging the order of summation and using (10)we obtain:

$$F(n) = \sum_{k=1}^{n}\left(-\frac{4}{\pi}\sum_{j=1}^{m}\frac{\cos(2j(2k-1))}{4j^2-1}+O\left(\frac{1}{m}\right)\right) \tag{12}$$

$$= -\frac{4}{\pi}\sum_{j=1}^{m}\frac{1}{4j^2-1}\sum_{k=1}^{n}\cos(2j(2k-1))+O\left(\frac{n}{m}\right) \tag{13}$$

$$= -\frac{4}{\pi}\sum_{j=1}^{m}\frac{\sin(4nj)}{(4j^2-1)2\sin(2j)}+O\left(\frac{n}{m}\right) \tag{14}$$

$$|F(n)| < \frac{4}{\pi}\sum_{j=1}^{m}\frac{1}{(4j^2-1)2|\sin(2j)|}+O\left(\frac{n}{m}\right) \tag{15}$$

From Hata's theorem(1992) we know that for P and Q natural we have, except for a finite number of cases:

$$|\pi - P/Q| > 1/Q^9 \tag{16}$$

$$|\sin(Q)| = O\left(1/Q^8\right) \tag{17}$$

Applying Hata's theorem to (15) and summing the series, we get that:

$$|F(n)| = O\left(m^7\right) + O\left(\frac{n}{m}\right) \tag{18}$$

Now there is a rather subtle trick from AoPS Math Forum. By subtle I mean I did not follow it at first. We can choose m in (18) for each n to get a tight bound on $|F(n)|$. We can do this because the truncation error is taken into account explicitly. So, take $m = O\left(n^{1/8}\right)$.

$$|F(n)| = O\left(n^{7/8}\right) \tag{19}$$

Applying the estimate from (19) to $F(n)$ and $F(k)$ in (6), the sum on the right of (6) converges absolutely.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k-1}\left(|\sin(2k-1)|-\frac{2}{\pi}\right) = L_{odd} \tag{20}$$

By similar arguments on G(n) we obtain:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k}\left(|\sin(2k)|-\frac{2}{\pi}\right) = L_{even} \tag{21}$$

Now if we subtract (20) from (21) and combine the indices, we get:

$$\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{(-1)^k|sin(k)|}{k} - \frac{2}{\pi}\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{(-1)^k}{k} = L_{even} - L_{odd} \tag{22}$$

$$\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{(-1)^k|sin(k)|}{k} = -\frac{2}{\pi}\ln(2) + L_{even} - L_{odd} \tag{23}$$

Therefore, the sum in (1) converges. ¡b00m!

dioid (IRC nick) has calculated 10^9 terms of this sum in groups of 10^3 and then applied the Levin-u transformation for acceleration and got -0.4050635483148443 as accelerated sum, error 0.0000000000000296. This is the best estimate so far of the famous dioid-landen constant.