landen's theorem. *The series*

$$S = \sum_{n=1}^{+\infty} n^{-2-\sin(n)} \tag{1}$$

diverges.

Proof. After an initial investigation we will bound a sequence S_m. The terms of S_m are subseries of (1) made with $L_{m\,j}$, an infinite subsequence of $\{1, 2, \ldots, +\infty\}$ with m constant and $j = 1, 2, \ldots, +\infty$:

$$S > S_m = \sum_{j=1}^{+\infty} L_{mj}^{-2-\sin(L_{m\,j})} \tag{2}$$

Then we will show that $\lim_{m\to\infty} S_m$ is unbounded and hence (1) diverges. To make a term of S_m we want to extract from (1) a subseries in which $\sin(n)$ is close to -1 in a known way.

We first find an integer r_m which has the property that $\sin(r_m)$ is close to -1. From the theory of continued fractions (see theorems 10 and 11 here) all the rational convergents of the continued fraction represention of any irrational number, α, $\{p_k/q_k : k \in \{0, 1, 2, \ldots, +\infty\}\}$, are "close" to α and satisfy these inequalities, giving lower and upper rational bounds on α:

$$\text{For } k \text{ even: } 0 < \alpha - p_k/q_k < 1/(q_k q_{k+1}). \tag{3}$$

$$\text{For } k \text{ odd: } 0 < p_k/q_k - \alpha < 1/(q_k q_{k+1}). \tag{4}$$

Let $\alpha = \pi/2$. Make an increasing sequence from all odd q_k. Successive q_k are relatively prime so there are infinitely many odd q_k. We select the m^{th} odd q_k from this sequence. Using the parity of k, multiply the applicable equation (3) or (4) by q_k:

$$|q_k\pi/2 - p_k| < 1/q_{k+1} \tag{5}$$

Taking the sin of both sides of (5), squaring, using trigonometric identities and applying Taylor's theorem approximation to bound the sine and the square root:

$$\begin{aligned}
\cos^2(p_k) &< \sin^2(1/q_{k+1})\\
1 - \cos^2(p_k) &> 1 - \sin^2(1/q_{k+1})\\
\sin^2(p_k) &> 1 - 1/{q_{k+1}}^2\\
|\sin(p_k)| &> 1 - 1/(2{q_{k+1}}^2)
\end{aligned}$$

Next we take r_m to be either p_k or $-p_k$ so that $\sin(r_m)$ is negative.

$$\sin(r_m) < -1 + 1/(2{q_{k+1}}^2) \tag{6}$$

Next we construct an upper and a lower approximation to $\pi/2$. From (5) we have an lower(upper) approximation p_k/q_k, depending on the parity of k, with p_{k+1}/q_{k+1} being respectively an upper(lower) approximation of higher accuracy.

$$|\pi/2 - p_k/q_k| < 1/(q_k q_{k+1}); \qquad |\pi/2 - p_{k+1}/q_{k+1}| < 1/(q_{k+1}q_{k+2}) \tag{7}$$

If we multiply the bounds in (7) by 4 we get an upper and a lower approximation to 2π. We can get integer approximations to a multiple of 2π, $\{4p_k, 4p_{k+1}\}$, by multiplying the first equation in (7) by q_k and the second equation by q_{k+1}. Define s_m to be the numerator of the lower rational approximation to 2π and t_m to be the numerator of the upper rational approximation to 2π. s_m and t_m will both have an error less that $4/q_{k+1}$, whether we multiplied the approximations to 2π by q_k or q_{k+1}.

Define for each m the sequence $L_{m\,j}$ of positive integers:

$$L_{m\,1} = \begin{cases} r_m + t_m & \text{if } r_m (\operatorname{mod} 2\pi) < 3\pi/2 \\ r_m + s_m & \text{if } r_m (\operatorname{mod} 2\pi) > 3\pi/2 \end{cases} \tag{8}$$

$$\text{Since } |r_m| < \max(s_m, t_m),\ L_{m\,1} < 2\max(s_m, t_m). \tag{9}$$

For terms in $L_{m\,j}$ with $j > 1$ define:

$$L_{m\,j} = \begin{cases} L_{m\,j-1} + t_m & \text{if } L_{m\,j-1} (\bmod 2\pi) < 3\pi/2 \\ L_{m\,j-1} + s_m & \text{if } L_{m\,j-1} (\bmod 2\pi) > 3\pi/2 \end{cases} \tag{10}$$

$$L_{m\,1} (\bmod 2\pi) = 3\pi/2 + u_1 \text{ with } |u_1| < 4/q_{k+1} \tag{11}$$

This is because the error in the choice of s_m or t_m is canceled in part by the error of opposite sign in r_m. Likewise, each succeeding $L_{m\,j}$ is steered by the choice of s_m or t_m to keep its $(\bmod 2\pi)$ close to $3\pi/2$ also. The errors do not grow:

$$L_{m\,j} (\bmod 2\pi) = 3\pi/2 + u_j \text{ with } |u_j| < 4/q_{k+1} \tag{12}$$

Next return to (2):

$$S_m = \sum_{j=1}^{+\infty} L_{m\,j}^{-2-\sin(L_{m\,j})} \tag{13}$$

Getting bounds on the exponent of (13)

$$2 + \sin(L_{m\,j}) = 2 - \cos(u_j) = 1 + 2\sin^2(u_j/2) \tag{14}$$

$$2 + \sin(L_{m\,j}) < 1 + 8/{q_{k+1}}^2 \tag{15}$$

The $L_{m\,j}$ are not equally spaced as j increases. However:

$$L_{m\,j+1} - L_{m\,j} < \max(s_m, t_m) \text{ and by induction}$$

$$L_{m\,j} < L_{m\,1} + (j-1)\max(s_m, t_m) < (j+1)\max(s_m, t_m), \text{ using (9).}$$

$$\max(s_m, t_m) \le 4p_{k+1} < 8q_{k+1}, \text{ using } \pi/2 < 2 \tag{16}$$

Combining these approximations we get:

$$S_m > \sum_{j=2}^{+\infty} (8q_{k+1}j)^{-1-8/{q_{k+1}}^2} \tag{17}$$

$$S_m > \int_{j=2}^{+\infty} (8q_{k+1}j)^{-1-8/{q_{k+1}}^2} dj = (8q_{k+1})^{(1-8/{q_{k+1}}^2)}/(2^{8/q_{k+1}}512) \tag{18}$$

As m increases, the values of q_{k+1} increase without bound and the lower bounds of S_m increase without bound so (1) diverges. ¡b00m!