Textbook Partial Fractions Problems

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Typical partial fractions problems would be ones like

$$\int \frac{x^3+4}{(x^2-1)(x^2+3x+2)}dx$$
$$\int \frac{x^3+2x-1}{(x^2-x-2)^2}dx$$
$$\int \frac{dx}{(x^6-1)^4}$$
$$\int \frac{dx}{(64x^4+81)^3}$$
$$\int \frac{dx}{(x^8-16)^2}$$

The partial fractions algorithm as taught is inferior. Computer algebra systems use superior methods, some of which are usable by humans, and others less so.

1 Reduction of Numerators

Assume throughout that $(n(x), d(x)) = 1$.
When $f(x) = \frac{n(x)}{d(x)}$ with $\deg(n) \geq \deg(d)$ a reduction in degree is immediate by polynomial division. If $n(x) = q(x)d(x) + r(x)$ then $\int \frac{n(x)}{d(x)}dx = \int q(x)dx + \int \frac{r(x)}{d(x)}dx$. Suppose that $\deg(n) = \deg(d) - 1$. Then $n(x) = Kd'(x) + r(x)$ for some constant K, and $\int \frac{n(x)}{d(x)}dx = \int \frac{r(x)}{d(x)}dx + K\log(d(x))$. So one may assume that $\deg(n) \leq \deg(d) - 2$.
Suppose $d(x)$ is even, so that $d(x) = d_e(x^2)$. Then $n(x) = xn_o(x^2) + n_e(x^2)$ and one may carry out the decomposition

$$\begin{aligned}\int \frac{n(x)}{d(x)}dx &= \int \frac{n_o(x^2)}{d_e(x^2)}xdx + \int \frac{n_e(x^2)}{d_e(x^2)}dx \\ &= \frac{1}{2}\int^{x^2} \frac{n_o(y)}{d_e(y)}dy + \int \frac{n_e(x^2)}{d_e(x^2)}dx\end{aligned}$$

Suppose $d(x)$ is odd, so that $d(x) = xd_o(x^2)$, and $n(x) = n_e(x^2) + xn_o(x^2)$. Then

$$\int \frac{n_e(x^2) + xn_o(x^2)}{xd_o(x^2)}dx = \frac{1}{2}\int^{x^2} \frac{n_e(y)}{d_o(y)}\frac{dy}{y} + \int \frac{n_o(x^2)}{d_o(x^2)}dx$$

More generally it's possible to reduce rational functions in a similar manner.

$$\begin{aligned}\int \frac{n(x)}{d(x)}dx &= \frac{1}{2}\int \left(\frac{n(x)}{d(x)} + \frac{n(-x)}{d(-x)}\right)dx + \frac{1}{2}\int \left(\frac{n(x)}{d(x)} - \frac{n(-x)}{d(-x)}\right)dx \\ &= \int R_e(x^2)dx + \frac{1}{2}\int^{x^2} R_o(y)dy\end{aligned}$$

where

$$\begin{aligned}R_e(x^2) &= \frac{1}{2}\left(\frac{n(x)}{d(x)} + \frac{n(-x)}{d(-x)}\right) \\ xR_o(x^2) &= \frac{1}{2}\left(\frac{n(x)}{d(x)} - \frac{n(-x)}{d(-x)}\right)\end{aligned}$$

but without some sort of simpler common denominator, this isn't often actually a simplification (though it may be when radicals become involved). Sometimes it can be useful when $d(x)$ is given in a factored form where the bulk of the degree is consumed by even factors, or when $d(x) = (ax^2 + bx + c)^n$ after substituting $y = x + \frac{b}{2a}$.

2 Reduction of Denominators to Squarefree

$d(x)$ is squarefree if $(d(x), d'(x)) = 1$; otherwise $d(x) = \prod d_k(x)^k$ with each $d_k(x)$ squarefree, which decomposition is called the *squarefree factorization*.
In that case $r(x) = (d(x), d'(x)) = \prod_{k\geq 2} d_k(x)^{k-1}$ so $q(x) = \frac{d(x)}{r(x)} = \prod d_k(x)$, and $(q(x), r(x)) = \prod_{k\geq 2} d_k(x)$ and finally

$$d_1(x) = \frac{d(x)}{r(x)(q(x), r(x))}$$

Applying this process recursively to $r(x)$ yields the remaining terms. Remembering $(q(x), r(x))$ at one stage to use it as $q(x)$ for the next may be useful.
If the denominator is not given in factored form, this process will determine what's necessary. If it is, merely group the factors appropriately. In either case, the squarefree factorization is available.
Let $d(x) = u(x)d_m(x)^m$ where $d_m(x)$ is the highest power divisor of $d(x)$ in its squarefree factorization, and $u(x)$ its associated quotient. Note that $(u(x)d'_m(x), d_m(x)) = 1$ and use the extended Euclidean algorithm to determine $a(x), b(x)$ so that

$$n(x) = a(x)u(x)d'_m(x) + b(x)d_m(x)$$

Dividing by $d(x)$ one obtains

$$\begin{aligned}
\frac{n(x)}{d(x)} &= \frac{a(x)d'_m(x)}{d_m(x)^m} + \frac{b(x)}{u(x)d_m(x)^{m-1}} \\
&= \frac{a(x)d'_m(x)}{d_m(x)^m} - \frac{1}{m-1}\cdot\frac{a'(x)d_m(x)}{d_m(x)^m} + \frac{1}{m-1}\cdot\frac{(m-1)b(x)+a'(x)u(x)}{u(x)d_m(x)^{m-1}} \\
&= -\frac{1}{m-1}\left(\frac{d}{dx}\,\frac{a(x)}{d_m(x)^{m-1}}\right) + \frac{1}{m-1}\cdot\frac{(m-1)b(x)+a'(x)u(x)}{u(x)d_m(x)^{m-1}} \\
\int\frac{n(x)}{d(x)}dx &= \frac{1}{m-1}\int\frac{(m-1)b(x)+a'(x)u(x)}{u(x)d_m(x)^{m-1}}dx - \frac{1}{m-1}\cdot\frac{a(x)}{d_m(x)^{m-1}}
\end{aligned}$$

which reduction may be applied recursively until $d(x)$ is squarefree.

3 Integration with Squarefree Denominators

Assume $d(x)$ has no rational roots or linear factors. If it has any, the corresponding factors are easily subtracted out: $\lim_{x\to a} \frac{n(x)}{d(x)}(x-a)$ provides the necessary coefficient.
All the reductions of the first section apply as well, so one may further assume $\deg(n) \leq \deg(d) - 2$ and evenness and oddness are all eliminated.
The Chinese Remainder Theorem supplies a decomposition of $\frac{n(x)}{d(x)}$ using the factors of $d(x)$ recorded during its squarefree factorization or otherwise supplied by the problem.

$$n(x) = \sum_k (n(x) \bmod d_k(x)) \cdot \left(\left(\frac{d(x)}{d_k(x)} \right)^{-1} \bmod d_k(x) \right) \frac{d(x)}{d_i(x)}$$

The mod denotes the least residue, and the inverse is the multiplicative inverse in the appropriate quotient group. For larger factors, this is more work than the matrix computations for traditional partial fractions. When the $d_k(x)$ are at most quadratic, the polynomials are appropriately small and divide and conquer strategies work well for computing products $\bmod d_k(x)$.
From here, it's largely hoped one has the full factorization. If not, more can be done, but it's not really usable by humans and involves computing resolvents and the like. Often textbook problems supply factorizations of denominators down to repeated quadratic and linear factors in $\mathbb{Q}[x]$.
Quadratic denominators uniformly fall down to the $\deg(n) = \deg(d) - 1$ reduction of the first section where not arctangents or logarithms of linear factors for irrational roots, so squarefree decompositions into quadratic factors suffice.

4 Examples

Example 1

$$\int \frac{x^3+4}{(x^2-1)(x^2+3x+2)}dx$$

Subtract out an appropriate linear combination of reciprocals of the highest powers of linear divisors of the denominator.

$$d(x) = (x-1)(x+2)(x+1)^2$$

$$\lim_{x\to 1} \frac{x^3+4}{d(x)}(x-1) = \frac{5}{2^2\cdot 3}$$

$$\lim_{x\to -1} \frac{x^3+4}{d(x)}(x+1)^2 = -\frac{3}{2}$$

$$\lim_{x\to -2} \frac{x^3+4}{d(x)}(x+2) = \frac{4}{3}$$

$$\begin{aligned}\frac{x^3+4}{(x-1)(x+2)(x+1)^2} &= \frac{5}{12}\cdot\frac{1}{x-1} + \frac{1}{12}\cdot\frac{7x^2-13x-38}{(x-2)(x+1)^2}\\ &= \frac{5}{12}\cdot\frac{1}{x-1} + \frac{4}{3}\cdot\frac{1}{x+2} - \frac{3}{4}\cdot\frac{x+3}{(x+1)^2}\\ &= \frac{5}{12}\cdot\frac{1}{x-1} + \frac{4}{3}\cdot\frac{1}{x+2} - \frac{3}{2}\cdot\frac{1}{(x+1)^2} - \frac{3}{4}\cdot\frac{1}{x+1}\end{aligned}$$

$$\int \frac{x^3+4}{(x-1)(x+2)(x+1)^2}dx = \frac{5}{12}\log(x-1) + \frac{3}{2}\cdot\frac{1}{x+1} + \frac{4}{3}\log(x+2) - \frac{3}{4}\log(x+1)$$

Note here that the limits are only used for the highest powers of a linear divisor of $d(x)$; the derivatives for other residues are too much work.

Example 2

$$\int \frac{x^2+2x-1}{(x^2-x-2)^2}dx$$

Use the recurrence to reduce the denominator to squarefree.

$$\begin{aligned}
x^3+2x-1 &= a(x)(2x-1)+b(x)(x^2-x-2)\\
x^3+2x-1 &\equiv \frac{1}{8} \mod 2x-1\\
&\equiv -\frac{9}{4}b(x) \mod 2x-1\\
b(x) &\equiv -\frac{1}{18} \mod 2x-1\\
x^3+2x-1 &\equiv 5x+1 \mod x^2-x-2\\
&\equiv (2x-1)a(x)\\
(5x+1)(2x-1) &\equiv 7x+19 \mod x^2-x-2\\
&\equiv (2x-1)^2a(x) \mod x^2-x-2\\
&\equiv (4x^2-4x-1)a(x) \mod x^2-x-2\\
&\equiv 9a(x) \mod x^2-x-2\\
x^3+2x-1 &= \left(\frac{7x+19}{9}+t(x^2-x-2)\right)(2x-1)-\frac{x^2-x-2}{18}\\
-1 &= \left(\frac{19}{9}-2t\right)(-1)-\frac{-2}{18}\\
t &= \frac{1}{2}\\
a(x) &= \frac{1}{2}x^2+\frac{5}{18}x+\frac{10}{9}\\
b(x) &= -\frac{1}{18}\\
x^3+2x-1 &= \left(\frac{1}{2}x^2+\frac{5}{18}x+\frac{10}{9}\right)(2x-1)-\frac{x^2-x-2}{18}\\
\int \frac{x^3+2x-1}{(x^2-x-2)^2}dx &= \int \frac{-\frac{1}{18}+(x+\frac{5}{18})}{x^2-x-2}dx-\frac{\frac{1}{2}x^2+\frac{5}{18}x+\frac{10}{9}}{x^2-x-2}\\
&= \frac{1}{9}\int \frac{9x+2}{x^2-x-2}dx-\frac{1}{18}\cdot\frac{9x^2+5x+20}{x^2-x-2}\\
9x+2 &= \frac{9}{2}(2x-1)+\frac{13}{2}\\
\int \frac{x^3+2x-1}{(x^2-x-2)^2}dx &= \frac{1}{2}\int \frac{2x-1}{x^2-x-2}dx+\frac{13}{18}\int \frac{1}{x^2-x-2}dx-\frac{1}{18}\cdot\frac{9x^2+5x+20}{x^2-x-2}\\
&= \frac{13}{18}\int \frac{1}{x^2-x-2}dx+\frac{1}{2}\log(x^2-x-2)-\frac{1}{18}\cdot\frac{9x^2+5x+20}{x^2-x-2}\\
&= \frac{1}{2}\log(x^2-x-2)-\frac{13}{27}\operatorname{arctanh}\left(\frac{2x-1}{3}\right)-\frac{1}{18}\cdot\frac{9x^2+5x+20}{x^2-x-2}
\end{aligned}$$

Alternatively, substitute $y = x - \frac{1}{2}$ and use evenness of the denominator.

$$\begin{aligned}
\int \frac{x^3+2x-1}{(x^2-x-2)^2}dx &= \int \frac{(y+\frac{1}{2})^3+2y}{((y+\frac{1}{2})^2-y-\frac{5}{2}}dy \\
&= 2\int^{x-\frac{1}{2}} \frac{8y^3+12y^2+22y+1}{(4y^2-9)^2}dy \\
&= 2\int^{x-\frac{1}{2}} \frac{4y^2+11}{(4y^2-9)^2}(2y)dy + 2\int^{x-\frac{1}{2}} \frac{12y^2+1}{(4y^2-9)^2}dy \\
&= 2\int^{\frac{1}{4}(2x-1)^2} \frac{4z+11}{(4z-9)^2}dz + 2\int^{x-\frac{1}{2}} \frac{12y^2+1}{(4y^2-9)^2}dy \\
&= 2\int^{\frac{1}{4}(2x-1)^2} \left(\frac{1}{4z-9}+\frac{20}{(4z-9)^2}\right)dz + 2\int^{x-\frac{1}{2}} \frac{12y^2+1}{(4y^2-9)^2}dy \\
&= 2\left(\frac{1}{4}\log(4z-9) - \frac{5}{4z-9}\right)\Bigg|_{z=\frac{1}{4}(2x-1)^2} + 2\int^{x-\frac{1}{2}} \frac{12y^2+1}{(4y^2-9)^2}dy \\
&= \frac{1}{2}\log(x^2-x-2) - \frac{5}{2(x^2-x-2)} + 2\int^{x-\frac{1}{2}} \frac{12y^2+1}{(4y^2-9)^2}dy \\
12y^2+1 &= 3(4y^2-9)+28 \\
\int \frac{x^3+2x-1}{(x^2-x-2)^2}dx &= \frac{1}{2}\log(x^2-x-2) - \frac{5}{2(x^2-x-2)} + \int^{x-\frac{1}{2}} \left(\frac{56}{(9-4y^2)^2} - \frac{6}{9-4y^2}\right)dy \\
y &= \frac{3}{2}\tanh\left(\frac{2}{3}z\right) \\
z &= \frac{3}{2}\operatorname{arctanh}\left(\frac{2x-1}{3}\right) \\
\int \frac{x^3+2x-1}{(x^2-x-2)^2}dx &= \frac{1}{2}\log(x^2-x-2) - \frac{5}{2(x^2-x-2)} + \int^{\frac{3}{2}\operatorname{arctanh}\left(\frac{2x-1}{3}\right)} \left(\frac{56}{81}\cosh\left(\frac{2}{3}z\right)^2 - \frac{2}{3}\right)dz \\
&= \frac{1}{2}\log(x^2-x-2) - \frac{5}{2(x^2-x-2)} + \left(\frac{14}{27}\sinh\left(\frac{4}{3}z\right) - \frac{26}{81}z\right)\Bigg|_{z=\frac{3}{2}\operatorname{arctanh}\left(\frac{2x-1}{3}\right)} \\
&= \frac{1}{2}\log(x^2-x-2) - \frac{5}{2(x^2-x-2)} + \frac{14}{27}\sinh\left(2\operatorname{arctanh}\left(\frac{2x-1}{3}\right)\right) \\
&\quad - \frac{13}{27}\operatorname{arctanh}\left(\frac{2x-1}{3}\right) \\
&= \frac{1}{2}\log(x^2-x-2) - \frac{5}{2(x^2-x-2)} - \frac{7}{18}\cdot\frac{2x-1}{x^2-x-2} - \frac{13}{27}\operatorname{arctanh}\left(\frac{2x-1}{3}\right) \\
&= \frac{1}{2}\log(x^2-x-2) - \frac{\frac{14}{9}x+\frac{5}{2}-\frac{7}{18}}{x^2-x-2} - \frac{13}{27}\operatorname{arctanh}\left(\frac{2x-1}{3}\right) \\
&= \frac{1}{2}\log(x^2-x-2) - \frac{1}{9}\cdot\frac{7x+19}{x^2-x-2} - \frac{13}{27}\operatorname{arctanh}\left(\frac{2x-1}{3}\right)
\end{aligned}$$

The two results differ by a constant.

Example 3

$$\int \frac{dx}{(x^6 - 1)^4}$$

Reduce the denominator to squarefree via the integral recurrence, then resolve into quadratic factors.

Example 4

$$\int \frac{dx}{(64x^4 + 81)^3}$$

Reduce the denominator to squarefree via the integral recurrence, then resolve into quadratic factors.

Example 5

$$\int \frac{dx}{(x^8 - 16)^2}$$

Reduce the denominator to squarefree via the integral recurrence, then resolve into quadratic factors.