Solving ODEs in exact form and finding integrating factors for linear & some non-linear differential equations by using the integrating factor to make the ODE exact. If you don't care much about stepping through the logic of why it works, jump to the bottom to see examples.

The types of problems being attempted here are of the form:

$$\frac{dy}{dx} = \frac{D(x,y)}{E(x,y)}$$

or

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

note: The 2nd form is not technically a valid way to manipulate a differential equation when solving it, but it works well as a nmemonic device.

In order to find an integrating factor to assist in solving this problem I'll re-state a theorem from multivariable calculus:

Clairaut's Theorem (paraphrased):
Given $f(x, y)$, where f_{xy} exists at a point $(x, y) = (a, b)$, and is continuous, $f_{xy} = f_{yx}$ at that point. In other words, the 2nd derivative w/respect to y then x or x then y are equal (as long as the usual conditions on differentiable stuff hold)

Given some function $\psi(x, y)$ (*psi*), using the chain rule:

$$\frac{\partial\psi}{\partial x} = \frac{d\psi}{dy}\frac{dy}{dx} = D(x, y) \rightarrow \frac{dy}{dx} = \frac{D(x,y)}{\frac{d\psi}{dy}(x,y)} = \frac{\psi_x(x,y)}{\psi_y(x,y)}$$

Examining the latter form and comparing to the original problem:

$$\frac{dy}{dx} = \frac{D(x,y)}{E(x,y)} \qquad \Leftrightarrow \qquad \begin{array}{l} \psi_y(x, y) = E(x, y) \\ \psi_x(x, y) = D(x, y) \end{array}$$

... if a function $\psi(x, y)$ exists that meets these conditions, then ψ is a solution to the original differential equation. But, we seek a solution in 2 variables, not 3; introducing ψ adds a 3rd [dependent] variable, therefore we'll deal with this by finding a level curve of ψ:

$$\psi(x, y) = c$$

... where c plays the part of a constant of integration. If ψ satisfies Clairaut's Theorem and $\psi_x = M$, $\psi_y = N$, then the ODE is said to be *exact*. The ODE is sometimes written as:

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

... though the only difference between this and the original form is a sign difference between either M & D, or N & E, which is accounted for if the ODE is re-written in standard form:

$$y' - \frac{D(x,y)}{E(x,y)} = 0 \qquad \rightarrow \qquad y' + \frac{M(x,y)}{N(x,y)} = 0$$

Here's the solution method if the ODE is exact (not going through the derivation here, that can be found elsewhere):

(1) $\psi(x, y) = \int M(x, y)\, dx + h(y) = c$ or $\psi(x, y) = \int N(x, y)\, dy + h(x) = c$

(2) $\frac{d}{dy} \int M(x, y)\, dx + h'(y) = N(x, y)$ or $\frac{d}{dx} \int N(x, y)\, dy + h'(x) = M(x, y)$

In either case for (2), solve for h then plug it back into (1).

The trouble is, most ODEs are not already in exact form:

$$M(x, y)\, dx + N(x, y)\, dy = 0$$
$$M_y(x, y) \neq N_x(x, y)$$

Multiply the original equation through by $\mu(x, y)$, then calculate the x and y derivatives as if checking to ensure the ODE is in exact form:

$$\mu(x, y)\, M(x, y)\, dx + \mu(x, y)\, N(x, y)\, dy = 0$$

$$[\mu(x, y)\, M(x, y)]_y = [\mu(x, y)\, N(x, y)]_x$$
$$\rightarrow \qquad \mu_y(x, y)\, M(x, y) + \mu(x, y)\, M_y(x, y) = \mu_x(x, y)\, N(x, y) + \mu(x, y)\, N_x(x, y)$$

This is a new differential equation in μ instead of y (to find more on this, lookup 'adjoint', sometimes denoted by a † (dagger) symbol). To draw a comparison, the adjoint behaves similar to the transpose of a matrix. In solving an ill conditioned matrix equation $A\, x = b$, where there is no 'perfect' solution, least squares may be used to solve $A^T A\, \hat{x} = A^T b$, which is very similar to what is done here:

$$A^T A\, \hat{x} = A^T b \qquad \Leftrightarrow \qquad \mu(x, y)\, y' = \mu(x, y)\, \frac{\psi_x(x,y)}{\psi_y(x,y)}$$
$$\rightarrow\ y' = \frac{M^*(x,y) = \mu(x,y)\, \psi_x(x,y)}{N^*(x,y) = \mu(x,y)\, \psi_y(x,y)}$$

Anyway, to solve the new differential equation, examine the PDE in μ and attempt to guess a form for $\mu(x, y)$ that makes the PDE easy to solve. Constants of integration are ignored since *any* solution μ will suffice.

Example:

$$(3\, x^2\, y + 2\, x\, y + y^3)\, dx + (x^2 + y^2)\, dy = 0$$
$$\mu_y(x, y)\, (3\, x^2\, y + 2\, x\, y + y^3) + \mu(x, y)\, (3\, x^2 + 2\, x + 3\, y^2) = \mu_x(x, y)\, (x^2 + y^2) + \mu(x, y)\, (2\, x)$$

Look through the equation and mentally picture what happens when μ is assumed to be a function of only y or x. If y, $\mu_x = 0$, leaving:

$$\mu(y)\,3\,(x^2+y^2) - \mu'(y)\,(y^3+3\,x^2\,y+2\,x\,y) = 0$$

... but there doesn't seem to be a way to eliminate the xs, meaning our assumption is false (x cannot exist in the equation or solution since it's an ODE in μ, and μ is a function of y only). So, try x, $\mu_y = 0$ and:

$$\mu(x)\,3\,(x^2+y^2) = \mu'(x)\,(x^2+y^2)$$
$$\frac{\mu'}{\mu} = 3 \rightarrow \mu = e^{3\,x}$$

Finally, check that it's in exact form:

$$[e^{3\,x}(3\,x^2\,y+2\,x\,y+y^3)]_y \overset{?}{=} [e^{3\,x}(x^2+y^2)]_x$$
$$3\,x^2\,e^{3\,x}+2\,x\,e^{3\,x}+3\,y^2\,e^{3\,x} \overset{?}{=} 3\,x^2\,e^{3\,x}+2\,x\,e^{3\,x}+3\,y^2\,e^{3\,x}\,\blacksquare$$

example:

$$\left(3\,x+\frac{6}{y}\right)dx+\left(\frac{x^2}{y}+3\,\frac{y}{x}\right)dy = 0$$
$$\mu_y(x,\,y)\left(3\,x+\frac{6}{y}\right)+\mu(x,\,y)\left(-\frac{6}{y^2}\right) = \mu_x(x,\,y)\left(\frac{x^2}{y}+3\,\frac{y}{x}\right)+\mu(x,\,y)\left(2\,\frac{x}{y}-3\,\frac{y}{x^2}\right)$$

Sometimes it's useful to require that $\mu = Q(t)$, where t is some function of x and y.

$$\mu'(t)\,t_y\left(3\,x+\frac{6}{y}\right)+\mu(t)\left(-\frac{6}{y^2}\right) = \mu'(t)\,t_x\left(\frac{x^2}{y}+3\,\frac{y}{x}\right)+\mu(t)\left(2\,\frac{x}{y}-3\,\frac{y}{x^2}\right)$$
$$\mu'\left(t_y\left(3\,x+\frac{6}{y}\right)-t_x\left(\frac{x^2}{y}+3\,\frac{y}{x}\right)\right) = \mu\left(2\,\frac{x}{y}-3\,\frac{y}{x^2}+\frac{6}{y^2}\right)$$

$$\frac{\mu'}{\mu} = \frac{2\,\frac{x}{y}-3\,\frac{y}{x^2}+\frac{6}{y^2}}{t_y\left(3\,x+\frac{6}{y}\right)-t_x\left(\frac{x^2}{y}+3\,\frac{y}{x}\right)}$$
$$= \frac{1}{x\,y}\;\frac{2\,x^3\,y-3\,y^3+6\,x^2}{t_y(3\,x^2\,y+6\,x)-t_x(x^3+3\,y^2)}$$

Sometimes, pure guess work (or luck) is needed. Comparing the numerator/denominator to see if it can be eliminated completely ($\frac{a}{a}=1$), the terms available look very similar, so try solving:

$$2\,x^3\,y-3\,y^3+6\,x^2 = t_y(3\,x^2\,y+6\,x)-t_x(x^3+3\,y^2)$$

The factor for t_y on the right is a factor of x away from having the same value as 2 terms on the left excepting $3\,x^2\,y$ which would be off by a constant factor. The factor for t_x happens to be off by a factor of y and a bit of fudging for constants as well. t_y could contribute the missing x, and t_x the missing y hinting that $t = x\,y$. Using this substitution the ratio becomes 1, leaving:

$\frac{\mu'}{\mu} = \frac{1}{x\,y} = \frac{1}{t} \quad \to \ln|\,\mu\,| = \ln|\,t\,| \;\Rightarrow \mu = t = x\,y$

Plugging this back in & checking for exactness:

$$[3\,x^2\,y + 6\,x]_y - [x^3 + 3\,y^2]_x \overset{?}{=} 0$$
$$3\,x^2 - 3\,x^2 = 0$$

From here it's just a matter of solving the ODE which should now be in exact form.

2nd order ODEs:

Same basic idea, though the ODEs are usually non-linear in x to keep things simple, and an integrating factor is assumed to be a function of x only to simplify things:

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$
$$\mu(x)\,P(x)\,y'' + \mu(x)\,Q(x)\,y' + \mu(x)\,R(x)\,y = 0$$

The equation $[P(x)\,y']' + [f(x)\,y]' = 0$ is in exact form. The equation with μ multiplied in must be put in exact form, so the goal is to find a function μ that makes this process [relatively] trivial. The form we're after is $[\mu(x)\,P(x)\,y']' + [f(x)\,y]' = 0$.

By equating coefficients between the two eqns and eliminating *f(x)*, μ is found to satisfy:

$$P\,\mu'' + 2\,(P' - Q)\,\mu' + (P'' - Q' + R)\,\mu = 0 \qquad (3)$$

In general, this is just as difficult to solve, though in some cases it works out well. μ was forced to be a function of x because P Q & R are functions of x only. (3) wouldn't necessarily hold if an integrating factor were needed that is a function of y or x & y.
Also, before any time is spent trying to solve the ODE in μ by finding its adjoint,
at least in the general 2nd order ODE given here, the adjoint of the adjoint is the original ODE.