Claim: For any positive numbers $a_1 \le a_2 \le \cdots \le a_n$,

$$a_1^{a_1} \cdots a_n^{a_n} \ge (a_1 \cdots a_n)^{(a_1+\cdots+a_n)/n}.$$

Proof: We solve this using an averaging argument. Taking logs and dividing both sides by $a_1 + \cdots + a_n$, it suffices to show that

$$p_1 X_1 + \cdots + p_n X_n \ge \frac{1}{n} X_1 + \cdots + \frac{1}{n} X_n,$$

where $p_i = \frac{a_i}{a_1+\cdots+a_n}$, and $X_i = \log a_i$. Since the a_i are non-decreasing, so are the p_i and the X_i. All we need show is that any such weighted average, where the weights increase as the numbers increase, is bounded below by the arithmetic mean:

Since $p_1 \le \cdots \le p_n$ and $p_1 + \cdots + p_n = 1$, by averaging we have that $p_1 \le 1/n$ and $p_n \ge 1/n$. Let $i \in \{1, 2, 3, \ldots, n-1\}$ be the largest index such that $p_i \le 1/n$. Note that $\sum_{j=1}^{i} (1/n - p_j) = \sum_{j=i+1}^{n} (p_j - 1/n)$, since subtracting the left from the right yields

$$\sum_{j=1}^{n} \left(\frac{1}{n} - p_j \right) = \sum_{j=1}^{n} \frac{1}{n} - \sum_{j=1}^{n} p_j = 1 - 1 = 0,$$

and both are nonnegative by our choice of i.

Therefore,

$$\begin{aligned}
p_1 X_1 + \cdots + p_n X_n &= p_1 X_1 + \cdots p_i X_i + \sum_{j=i+1}^{n} \left(\left(p_j - \frac{1}{n} \right) X_j + \frac{1}{n} X_j \right) \\
&\ge p_1 X_1 + \cdots p_i X_i + \sum_{j=i+1}^{n} \left(\left(p_j - \frac{1}{n} \right) X_{i+1} \right) + \sum_{j=i+1}^{n} \frac{1}{n} X_j \\
&= p_1 X_1 + \cdots p_i X_i + \sum_{j=1}^{i} \left(\left(\frac{1}{n} - p_j \right) X_{i+1} \right) + \sum_{j=i+1}^{n} \frac{1}{n} X_j \\
&= \sum_{j=1}^{i} \left(p_j X_j + \left(\frac{1}{n} - p_j \right) X_{i+1} \right) + \sum_{j=i+1}^{n} \frac{1}{n} X_j \\
&\ge \sum_{j=1}^{i} \left(p_j X_j + \left(\frac{1}{n} - p_j \right) X_j \right) + \sum_{j=i+1}^{n} \frac{1}{n} X_j \\
&= \sum_{j=1}^{i} \frac{1}{n} X_j + \sum_{j=i+1}^{n} \frac{1}{n} X_j \\
&= \sum_{j=1}^{n} \frac{1}{n} X_j. \ \blacksquare
\end{aligned}$$