Ramanujan investigated nested radicals. His notebooks may contain proofs of the results below and if so they may be easier to follow.

Consider a sequence formed of radicals, $\sqrt{\ }$, nested n deep:

$$S_n = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}} \tag{1}$$

S_n satisfies the recurrence:

$$S_1 = 1; \quad S_{n+1} = \sqrt{1 + S_n} \tag{2}$$

We can show that $\lim_{n\to\infty} S_n$ exists by induction, proving that it is monotonically increasing and bounded above. By squaring equation (2) and manipulating:

$$\begin{aligned}
S_2 - S_1 &> 0 && (3)\\
\text{assume: } S_{n+1} - Sn &> 0 && (4)\\
S_{n+2}^2 - S_{n+1}^2 &= S_{n+1} - Sn && (5)\\
S_{n+2} - S_{n+1} &> 0 \text{ and by induction } S_n \text{ is monotonically increasing} && (6)\\
S_1 &< 2 && (7)\\
\text{assume: } S_n &< 2 && (8)\\
S_{n+1}^2 &= 1 + S_n < 1 + 2 < 4 && (9)\\
S_{n+1} &< 2 && (10)
\end{aligned}$$

Applying the results (6) and (10) with Lemma 1 from this proof of the Bolzano-Weierstrass theorem we have that S_n converges to a limit between 1 and 2. Call this limit ρ. Since $f(x) = \sqrt{1+x}$ is continuous,

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \sqrt{1 + S_n} && (11)\\
&= \sqrt{1 + \lim_{n\to\infty} S_n} && (12)\\
&= \sqrt{1 + \rho} && (13)\\
&= \frac{1 + \sqrt{5}}{2} && (14)
\end{aligned}$$

Does the following sequence have a limit and what is it?

$$R_n = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{\ldots\sqrt{1 + n\sqrt{1 + n + 1}}}}}}} \tag{15}$$

Consider more general sequences formed by the iterated compositions of radicals:

$$S_{k\,n} = \sqrt{1 + k\sqrt{1 + (k+1)\ldots\sqrt{1 + n\sqrt{1 + n + 1}}}} \tag{16}$$

and:

$$T_{k\,n} = \sqrt{1 + k\sqrt{1 + (k+1)\ldots\sqrt{1 + n\sqrt{1 + (n+1)(n+3)}}}} \tag{17}$$

Notice that $0 < S_{k\,n} < S_{k\,n}$ and that $S_{k\,n} < T_{k\,n}$. $T_{k\,n}$ has the fun property that $T_{k\,n} = k + 1$ for all n. Therefore, $\lim_{n\to\infty} S_{k\,n} \leq k + 1$ and exists since $S_{k\,n}$ is nondecreasing. Define $S_k = \lim_{n\to\infty} S_{k\,n}$. Note that $S_k \leq k + 1$, which will be needed later.

A calculation of $S_{2\,75} = 2.99999999999999999999976717\ldots$ suggests that $\lim_{n\to\infty} S_{k\,n} = k+1$. So we know what to prove.

Since the limits exist we can write:

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\begin{aligned}
S_k &> \sqrt{k\sqrt{k\sqrt{k\ldots}}} && (18)\\
S_k &> k^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots} && (19)\\
S_k &> k && (20)\\
k+1-S_k &= k+1-\sqrt{1+kS_{k+1}} && (21)\\
& \text{rationalizing on the right} && (22)\\
k+1-S_k &= \frac{(k+1)^2-1-kS_{k+1}}{k+1+\sqrt{1+(k+1)S_{k+1}}} && (23)\\
k+1-S_k &= \frac{k}{k+1+S_k}(k+2-S_{k+1}) && (24)\\
& \text{using } S_k > k \text{ and } S_{k+1} > k+1 \text{ on the right} && (25)\\
k+2-S_{k+1} &< 1 && (26)\\
k+1+S_k &> 2k+1 && (27)\\
k+1-S_k &< k/(2k+1) < \frac{1}{2} && (28)\\
S_k &> k+1/2 && (29)\\
& \text{Repeating these steps with the new lower bound } S_k > k+1/2 \text{ gives:} && (30)\\
S_k &> k+1/2+1/4\ldots \text{ etc.} && (31)\\
S_k &\leq k+1 \text{ as noted above} && (32)\\
S_k &= k+1 \text{ by sandwiching} && (33)
\end{aligned}
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