Lemma.
If k is a field, $f \in k[x]$ irreducible and $g \in k[x]$, then every irreducible factor of $f(g(x)) \in k[x]$ has degree divisible by $\deg f$. In particular, the number of irreducible factors of $f(g(x))$ is at most $\deg g$.

Pf.
If α is a root of an irreducible factor $p(x)$ of $f(g(x))$ then we have field extensions: $k(\alpha) \supseteq k(g(\alpha)) \supseteq k$. So $\deg p = [k(\alpha) : k] = [k(\alpha) : k(g(\alpha))][k(g(\alpha)) : k] = [k(\alpha) : k(g(\alpha))] \cdot \deg f$, since $g(\alpha)$ is a root of f. Lemma follows.

Solution to the Bonus Problem.
If $(x^2 + x)^{2^n} + 1 = r(x)s(x)$ for some $r, s \in \mathbb{Q}[x]$ then by the above lemma r and s are irreducible and have degree 2^n. We can assume that r, s are monic. Note that $(-1-x)^2 + (-1-x) = x^2 + x$, so $r(x)s(x) = r(-1-x)s(-1-x)$. By irreducibility and because both r, s have even degree, either $r(x) = r(-1-x)$ and $s(x) = s(-1-x)$, or $r(x) = s(-1-x)$ and $s(x) = r(-1-x)$. In the latter case, $(x^2 + x)^{2^n} + 1 = r(x)r(-1-x)$ and setting $x = -1/2$, the RHS is a square of a rational number whereas the LHS clearly isn't. In the first case, r and s are both polynomials in $x^2 + x$. Since $x^2 + x$ is transcendental over $\mathbb{Q}$, and by irreducibility of $x^{2^n} + 1 \in \mathbb{Q}[x]$, we must have either r or s equal to $(x^2 + x)^{2^n} + 1$.

To see that $r(x) = r(-1-x)$ implies $\exists p(x) \in \mathbb{Q}[x]$ such that $r(x) = p(x^2+x)$, note first that $r(0) = r(-1)$, so $r(x) - r(0)$ is divisible by $x(x+1) = x^2+x$. So $r(x) = a_0 + (x^2+x)r_1(x)$ for some $r_1 \in \mathbb{Q}[x]$. Now $a_0 + (x^2+x)r_1(-1-x) = r(-1-x) = r(x) = a_0 + (x^2+x)r_1(x)$ so that $r_1(-1-x) = r_1(x)$. So $r_1 = a_1 + (x^2+x)r_2$ for some $r_2 \in \mathbb{Q}[x]$. Proceed inductively (on the degree).